Dynamics of Electrically Coupled Harmonic Oscillators

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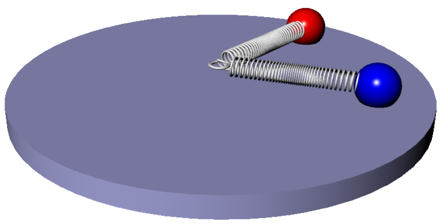
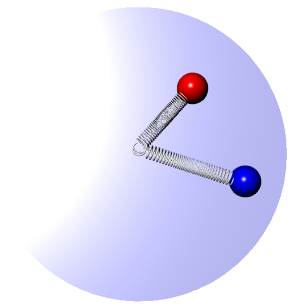
Motivation and Overview

We decided to take this course to understand the behavior of most basic systems that we could think of or would come across in our work. Now that we have learned how to analyze them and have developed an intuition for them, we wanted to put our new knowledge and skills to the test. We decided to derive our own system after coming across the harmonic oscillator problem in one of our problem sets. Deriving a new system would not only test our skill but also force us to incorporate the knowledge we gained in our physics and engineering courses over the last three years - a rather nice culmination of our undergraduate education.

The system we derived and wanted to understand is a system of two charged masses each attached to a center point by a spring (see figures below). In this report, we will explain how we derived this system from basic physics principles and how we tested our model to make sure it did not exhibit unrealistic behaviors. We then analyze the system by first changing the dimensional system into a dimensionless one. We do this because a dimensionless system makes our analysis simpler by looking at the ratios of parameters (more on this later). Finally, we find the fixed points, attempt to characterize them, and understand when the system becomes choatic. The challenge turned out harder than we expected but the effort was rewarding.





Another way of looking at the behavior of a two-body system like ours is to analyze the system from the perspective of one body. In their book on dynamical systems, Hirsch, Smale, and Devaney (HSB) refer to the system that is similar to ours as the gravitational two-body problem. The gravitational two-body problem is the system of two particles interacting with each other through gravity. The way HSB sets up the system for analysis is that they consider the system from the perspective of one of the bodies. The approach reduces the number of parameters of the system to make the analysis more manageable. However, HSB's approach does not consider the case of repelling force of particles. Their model assumes that the particles are not attracted to the middle of the system by some spring force. Therefore, a repelling force will cause the particles to move away from each other indefinitely, so the HSB model is not appropriate for the problem that our system solves. The major difference between the two models is the inclusion of springs that keep the particles a certain distance away from the center. Nevertheless, the approach of constructing the set of equations in terms of one particle relative to the other would have been an interesting system to model.

Governing Physics Equations

In order to derive the differential equations that governed the dynamics of this mechanical system, we began by analyzing the basic laws of physics behind the dynamics. The physics behind our system were comprised of summing forces and torques to determine translational and rotational accelerations.

Translational Accelerations

$$\sum F_1 = m_1 a_{r1} = F_{spring_1} + F_{elec_1} \bullet \hat{r_1} + F_{centripetal_1}$$

$$\sum F_2 = m_2 a_{r2} = F_{spring_2} + F_{elec_2} \bullet \hat{r_2} + F_{centripetal_2}$$

These equations reflect the idea that the sum of the forces in the radial direction on both masses will dictate the radial acceleration of the masses. Since the forces of the spring and centripetal motion are always in the radial direction, these forces are always purely in these directions. However, since the electric force depends on the relative location of the masses, the radial force must be calculated calculating the component of the electric force that is along the radial direction.

Rotational Accelerations

$$\frac{dL_1}{dt} = \sum \tau_1 = \tau_{elec_1} = F_{elec_1} \times r_1$$

$$\frac{dL_2}{dt} = \sum \tau_2 = \tau_{elec_2} = F_{elec_2} \times r_2$$

These equations reflect the idea that the change in angular momentum is equivalent to the sum of the torques on the individual masses. Since the only possible torque in the system can come from the electric field, it is the only component we account for in these equations. We can also see that the torque is calculated as the cross product between the electric force and the lever arm that is the length of r.

Governing Differential Equations

These equations can be written in the form of 4 coupled 2nd-order differential equations as follows:

1. $m_1\ddot{r_1} + K_1(r_1 - R_1) = K_e q_1 q_2 \left[\frac{r_1 - r_2 cos(\theta_1 - \theta_2)}{(r_1^2 - 2r_1 r_2 cos(\theta_1 - \theta_2) + r_2^2)^{(\frac{3}{2})}}\right] + m_1 r_1 (\dot{\theta_1})^2$

2. $m_1 r_1^2 \ddot{\theta_1} = K_e q_1 q_2 \left[\frac{r_1 r_2 sin(\theta_1 - \theta_2)}{(r_1^2 - 2r_1 r_2 cos(\theta_1 - \theta_2) + r_2^2)^{\frac{3}{2}}}\right] - 2m_1(\dot{\theta_1}) r_1 (\dot{r}_1)$

3. $m_2\ddot{r_2} + K_2(r_2 - R_2) = K_e q_1 q_2 \left[\frac{r_2 - r_1 cos(\theta_1 - \theta_2)}{(r_1^2 - 2r_1 r_2 cos(\theta_1 - \theta_2) + r_2^2)^{(\frac{3}{2})}}\right] + m_2 r_2 (\dot{\theta_2})^2$

4. $m_2 r_2^2 \ddot{\theta_2} = -K_e q_1 q_2 \left[\frac{r_1 r_2 sin(\theta_2 - \theta_1)}{(r_1^2 - 2r_1 r_2 cos(\theta_1 - \theta_2) + r_2^2)^{\frac{3}{2}}}\right] - 2m_2(\dot{\theta_1}) r_2 (\dot{r_2})$

The above equations can then be re-written in the form of a 8 first-order differential equations as follows:

$$r_1 = x_1 \quad \dot{r_1} = x_2 \quad \theta_1 = x_3 \quad \dot{\theta_1} = x_4 \quad r_2 = x_5 \quad \dot{r_2} = x_6 \quad \theta_2 = x_7 \quad \dot{\theta_2} = x_8$$

1. $\dot{x_1} = x_2$

2. $\dot{x_2} = \frac{K_e q_1 q_2}{m_1} \left[\frac{x_1 - x_5 cos(x_3 - x_7)}{(x_1^2 - 2x_1 x_5 cos(x_3 - x_7) + x_5^2)^{\frac{3}{2}}}\right] + x_1 x_4^2 - \frac{K_1}{m_1}(x_1 - R_1)$

3. $\dot{x_3} = x_4$

4. $\dot{x_4} = \frac{K_e q_1 q_2}{m_1 x_1^2} \left[\frac{x_1 x_5 sin(x_3 - x_7)}{(x_1^2 - 2x_1 x_5 cos(x_3 - x_7) + x_5^2)^{\frac{3}{2}}}\right] - 2\frac{x_2 x_4}{x_1}$

5. $\dot{x_5} = x_6$

6. $\dot{x_6} = \frac{K_e q_1 q_2}{m_2} \left[\frac{x_5 - x_1 cos(x_3 - x_7)}{(x_1^2 - 2x_1 x_5 cos(x_3 - x_7) + x_5^2)^{\frac{3}{2}}}\right] + x_5 x_8^2 - \frac{K_2}{m_2}(x_5 - R_2)$

7. $\dot{x_7} = x_8$

8. $\dot{x_8} = \frac{K_e q_1 q_2}{m_2 x_5^2} \left[\frac{x_1 x_5 sin(x_7 - x_3)}{(x_1^2 - 2x_1 x_5 cos(x_3 - x_7) + x_5^2)^{\frac{3}{2}}}\right] - 2\frac{x_6 x_8}{x_5}$

Validation of Equations

Due to the fact that these equations were derived from the basic physical laws, we desired to verify that our system was truly depicting the physical phenomenon. Since we understood our system consisted of a harmonic oscillator, we knew that there must be both conservation of energy as well as conservation of angular momentum. However, when we began our analysis, we primarily focused on conservation of energy. The result of these simulations informed us that we would also need to examine the conservation of angular momentum principles.

Dimensional Energy

The Equations

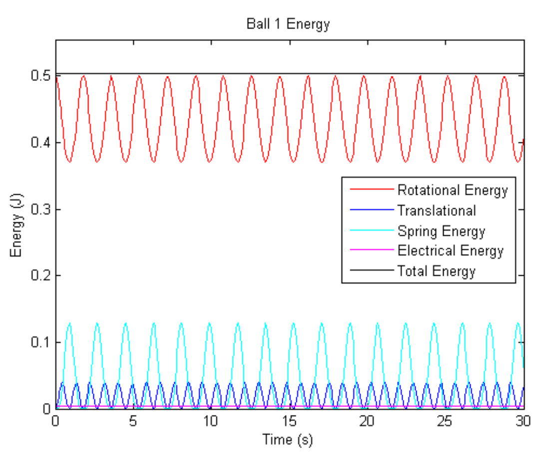
In order to analyze the total energy within our system, the following equations were derived in order to capture energy stored in rotational motion, translational motion, the electric field, and the spring.

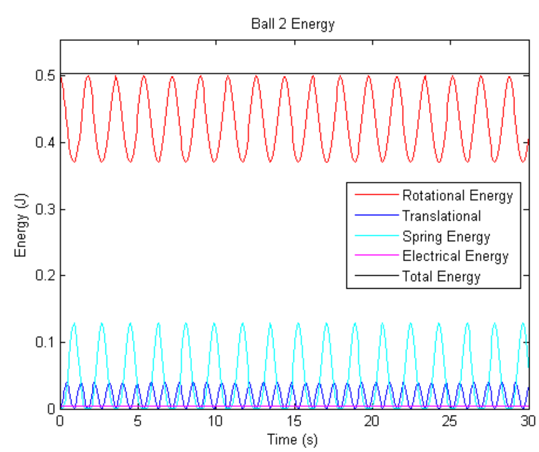
1. $e_{1rot} = \frac{1}{2}m_1 x_1^2 x_4^2$
2. $e_{1trans} = \frac{1}{2}m_1 x_2^2$
3. $e_{1spring} = \frac{1}{2}K_1(x_1 - R_1)^2$
4. $e_{1elec} = \frac{Keq_1q_2}{(x_1^2 - 2x_1x_5cos(x_3 - x_7) + x_5^2)^{\frac{1}{2}}}$
5. $e_{2rot} = \frac{1}{2}m_2 x_5^2 x_8^2$
6. $e_{2trans} = \frac{1}{2}m_2 x_2^2$
7. $e_{2spring} = \frac{1}{2}K_2(x_5 - R_2)^2$
8. $e_{2elec} = \frac{Keq_1q_2}{(x_1^2 - 2x_1x_5cos(x_3 - x_7) + x_5^2)^{\frac{1}{2}}}$

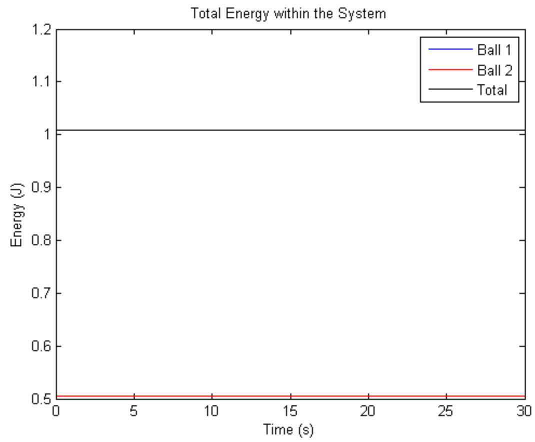
The Analysis

In our first simulations, we monitored the amount of energy in the system as it was allocated between the various components within the system. Our results provided interesting insights into the dynamics of our system. In order to understand these dynamics, we will examine two cases: (1) where the masses remain relatively far apart; (2) where the masses incur collisions.

In the first case (results below), we can see that energy seems to be relatively conserved between both masses. The total energy in the system does not vary and their is minimal energy transfer between the masses as they never come close to collisions.



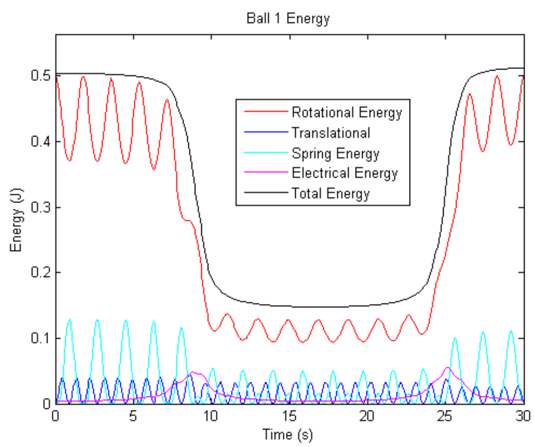


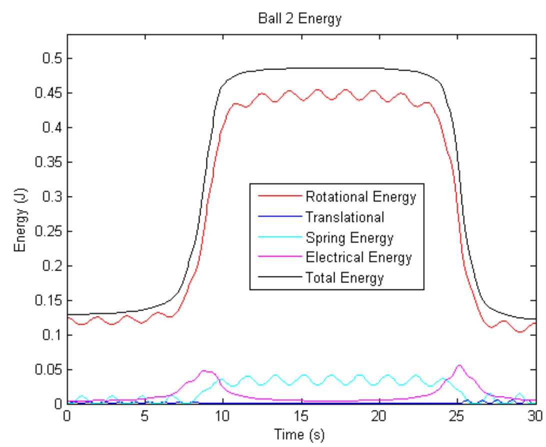

Total Energy within the System
Ball 1
Ball 2
Total
Energy (J)
Time (s)
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0
5
10
15
20
25
30

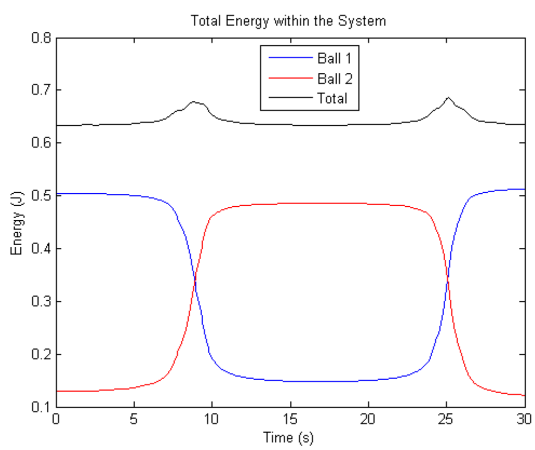
In the second case (results below), when we allow our masses to collide and transfer energy, an interesting phenomenon is observed. The total energy in the system appears to peak whenever the masses come very close to each other and have a transfer of energy. This can be explained by the fact that as two charges come very close, the force approaches a value close to infinity due to Coulomb's Law:

$$F = K_e \frac{q_1 q_2}{r^2}$$

Therefore, as the masses come very close, numerical inaccuracies in calculating the force become magnified and there appears to be a peak in the energy in the system. However, due to the fact that these peaks occur over very small time-steps, the overall energy in the system seems unaffected. However, this phenomenon in the numerical simulations pushed us to find another method of validation.







Dimensional Angular Momentum

In conducting our simulations and monitoring the energy in our system, we realized that by examining the total angular momentum in the system, we could avoid the numerical inaccuracies in monitoring the energy in the system. This can be explained by the fact that in calculating the angular momentum, there are no terms that approach infinity as the masses approach each other in a "collision."

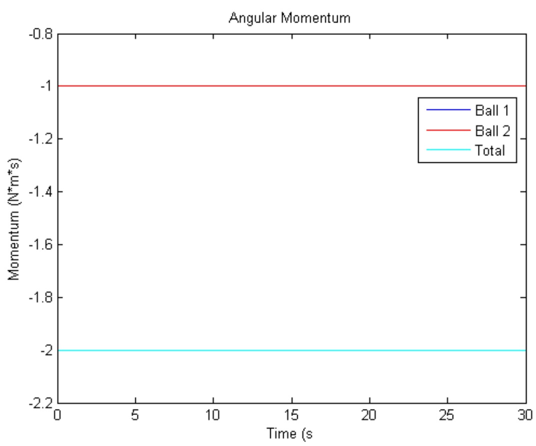
The Equations

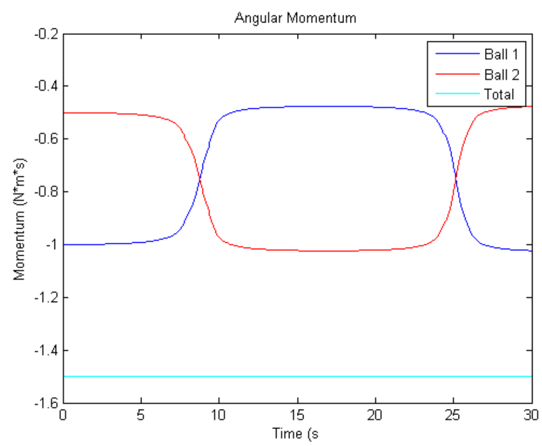
1. $l_1 = m_1 x_1^2 x_4$
2. $l_2 = m_2 x_5^2 x_8$

The Analysis

In examining the total angular momentum in the system, we first realized that as the laws of physics would mandate, angular momentum was conserved. We also noticed that even with collisions in the system, the total angular momentum did not exhibit the peak-like behavior as with the energy analysis. We can see this by examining the two same cases as before, while examining the total angular momentum in the system as oppose to the total energy.

As we can see in the results below, even when there are collisions in the system, the angular momentum remains constant and there are no numerical artifacts as in the energy simulations.





Dimensional Analysis

A challenge physicists and engineers encounter often when analyzing physical systems is isolating the parameters that have the most influence on the behavior of the system. Therefore, a common tool employed to understand the factors that affect a physical system is dimensional analysis. With this tool, the equations are scaled over constant values in the system in order to make the equations dimensionless. The resulting analysis yields equations with fewer parameters of constants but with more meaning. These parameters may be ratios or quantities that relate various forces or characteristics within the system. For example, the Reynold's number in fluid dynamics is a dimensionless number that gives a measure of the ratio between the inertial and viscous forces in a fluids system.

Therefore, in order to simplify our original dimensional equations that contained 9 dimensional parameters, we employed dimensional analysis to reduce these 9 parameters to 4 meaningful dimensionless parameters. In creating these parameters, we also needed to scale the variables and initial conditions.

Dimensionless Variables

In order to begin the dimensional analysis, we first scaled the dimensional variables within our equations. It is important to note here that X_3 and X_7 are not scaled as they are in units of radians that are inherently dimensionless.

$$T = \frac{t}{\sqrt{\frac{m_1}{K_1}}} \quad X_1 = \frac{x_1}{R_1} \quad X_2 = \frac{x_2}{\frac{R_1}{\sqrt{\frac{m_1}{K_1}}}} \quad X_4 = \frac{x_4}{\frac{1}{\sqrt{\frac{m_1}{K_1}}}}$$

$$X_5 = \frac{x_5}{R_1} \quad X_6 = \frac{x_6}{\frac{R_1}{\sqrt{\frac{m_1}{K_1}}}} \quad X_8 = \frac{x_8}{\frac{1}{\sqrt{\frac{m_1}{K_1}}}}$$

Dimensionless Parameters

In working through the dimensionless analysis, the following parameters were grouped together in order to assemble the following four parameters. Each of these parameters represents a certain important ratio in the dynamics of the system.

$$\mu = \frac{K_e q_1 q_2}{K_1 R_1^3} \quad \delta = \frac{R_1}{R_2} \quad \lambda = \frac{m_1}{m_2} \quad \zeta = \frac{K_1}{K_2}$$

Our first dimensionless parameter, μ, represents the ratio of electric forces to spring forces. The greater the absolute value of μ, the stronger the electric charges are and the more the behavior of the system is dominated by the electric field forces. It is also important to note that the sign of μ corresponds to whether the charges in the system are attractive or repulsive. Ultimately μ can have any real value, positive, negative, or zero. In the analysis of fixed points, this parameter proved to be very important in changing the qualitative behavior of the system.

Our remaining three dimensionless parameters relate the relative equilibrium spring lengths, the relative masses, and the relative spring constants. While these parameters are important to understanding the relative dynamics of the system, they are less influential on the qualitative behavior of the system. It is also important to note that all these have values that are only positive real numbers due to their physical meanings. In our analysis, we set these three parameters equal to 1 signifying the ratio of the masses, their spring constants, and their equilibrium lengths were all equal.

These dimensionless parameters and variables allowed us to derive the following dimensionless ODE's, initial conditions along with their corresponding dimensionless energy and angular momentum equations.

Dimensionless Equations

1. $\frac{dX_1}{dT} = X_2$
2. $\frac{dX_2}{dT} = \mu \left[\frac{X_1 - X_5 cos(X_3 - X_7)}{(X_1^2 - 2X_1 X_5 cos(X_3 - X_7) + X_5^2)^{\frac{3}{2}}} \right] + X_1 X_4^2 - (X_1 - 1)$
3. $\frac{dX_3}{dT} = X_4$
4. $\frac{dX_4}{dT} = \frac{\mu}{X_1^2} \left[\frac{X_1 X_5 sin(X_3 - X_7)}{(X_1^2 - 2X_1 X_5 cos(X_3 - X_7) + X_5^2)^{\frac{3}{2}}} \right] - 2\frac{X_2 X_4}{X_1}$
5. $\frac{dX_5}{dT} = X_6$
6. $\frac{dX_6}{dT} = \mu\lambda \left[\frac{X_5 - X_1 cos(X_3 - X_7)}{(X_1^2 - 2X_1 X_5 cos(X_3 - X_7) + X_5^2)^{\frac{3}{2}}} \right] + X_5 X_8^2 - \frac{\lambda}{\zeta}(X_5 - \frac{1}{\delta})$
7. $\frac{dX_7}{dT} = X_8$
8. $\frac{dX_8}{dT} = \frac{\mu\lambda}{X_5^2} \left[\frac{X_1 X_5 sin(X_7 - X_3)}{(X_1^2 - 2X_1 X_5 cos(X_3 - X_7) + X_5^2)^{\frac{3}{2}}} \right] - 2\frac{X_6 X_8}{X_5}$

Dimensionless Initial Conditions

1. $X_1(0) = \frac{x_{1,0}}{R_1}$
2. $X_2(0) = \frac{\frac{x_{2,0}}{R_1}}{\sqrt{\frac{m_1}{K_1}}}$
3. $X_3(0) = x_{3,0}$
4. $X_4(0) = \frac{\frac{x_{4,0}}{1}}{\sqrt{\frac{m_1}{K_1}}}$
5. $X_5(0) = \frac{x_{5,0}}{R_1}$
6. $X_6(0) = \frac{\frac{x_{6,0}}{R_1}}{\sqrt{\frac{m_1}{K_1}}}$
7. $X_7(0) = x_{7,0}$
8. $X_8(0) = \frac{\frac{x_{8,0}}{1}}{\sqrt{\frac{m_1}{K_1}}}$

Dimensionless Energy

1. $E_1 = X_1^2 X_4^2 + X_2^2 + (X_1 - 1)^2 + \frac{2\mu}{(X_1^2 - 2X_1 X_5 cos(X_3 - X_7) + X_5^2)^{\frac{1}{2}}}$
2. $E_2 = \frac{X_5^2 X_8^2}{\lambda} + \frac{X_6^2}{\lambda} + \frac{(X_5 - \frac{1}{\delta})^2}{\zeta} + \frac{2\mu}{(X_1^2 - 2X_1 X_5 cos(X_3 - X_7) + X_5^2)^{\frac{1}{2}}}$

Dimensionless Angular Momentum

1. $L_1 = X_1^2 X_4$
2. $L_2 = \frac{X_5^2 X_8}{\lambda}$

Fixed-Point Analysis

Fixed-Point Identification

Now that we have a simpler model, we can find the fixed points of the dimensionless system. The fixed points we find can be converted back into the dimensional system easily, so nothing is lost by analyzing the dimensionless system. To find the fixed points of the system, we set all equations to zero and we get the following:

1. $0 = X_2$
2. $0 = \mu\left[\frac{X_1 - X_5 cos(X_3 - X_7)}{(X_1^2 - 2X_1X_5cos(X_3-X_7)+X_5^2)^{\frac{3}{2}}}\right] + X_1X_4^2 - (X_1 - 1)$
3. $0 = X_4$
4. $0 = \frac{\mu}{X_1^2}\left[\frac{X_1X_5sin(X_3 - X_7)}{(X_1^2 - 2X_1X_5cos(X_3-X_7)+X_5^2)^{\frac{3}{2}}}\right] - 2\frac{X_2X_4}{X_1}$
5. $0 = X_6$
6. $0 = \mu\left[\frac{X_5 - X_1 cos(X_3 - X_7)}{(X_1^2 - 2X_1X_5cos(X_3-X_7)+X_5^2)^{\frac{3}{2}}}\right] + X_5X_8^2 - (X_5 - 1)$
7. $0 = X_8$
8. $0 = \frac{\mu}{X_5^2}\left[\frac{X_1X_5sin(X_7 - X_3)}{(X_1^2 - 2X_1X_5cos(X_3-X_7)+X_5^2)^{\frac{3}{2}}}\right] - 2\frac{X_6X_8}{X_5}$

Notice that the last terms in $\dot{x}_4$ and $\dot{x}_8$are zero because $x_4$and x_8 are zero. Therefore, for $\dot{x}_4$ and $\dot{x}_8$to equal zero either x_1, x_5, or$|sin(x_3 - x_7)|$must equal zero. We know that $x_1$and x_5 cannot equal zero because they are in the denominators of terms in $x_4$and x_8. Thus, $|sin(x_3 - x_7)| = 0$. Now, solving $|sin(x_3 - x_7)| = 0$, we find that $x_3 - x_7 = 0$ or multiples of π, but it is enough to consider 0 andπ. With this analysis, we can simplify the above equations to the following:

1. $0 = \mu\left[\frac{X_1 - X_5 cos(X_3 - X_7)}{(X_1^2 - 2X_1X_5cos(X_3-X_7)+X_5^2)^{\frac{3}{2}}}\right] - (X_1 - 1)$
2. $0 = \mu\left[\frac{X_5 - X_1 cos(X_3 - X_7)}{(X_1^2 - 2X_1X_5cos(X_3-X_7)+X_5^2)^{\frac{3}{2}}}\right] - (X_5 - 1)$

With that, we find that $cos(x_3 - x_7) = -1$ or 1. We now find the fixed points for two different cases.

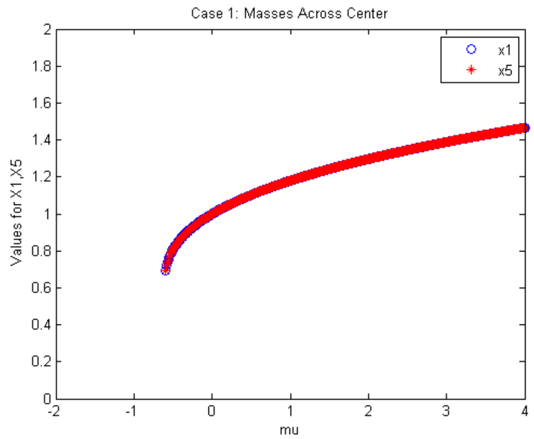
Case 1

In our first case, we are examining the case for when the charges are π radians apart or across each other from the center of their orbits. For $cos(x_3 - x_7) = -1$, we find that $x_1 = x_5$ and get the following two equations after plugging in $x_1 = x_5$:

$$0 = \frac{\mu\left|\frac{1}{x_5}\right|}{4x_5} - x_5 + 1$$

$$0 = \frac{\mu\left|\frac{1}{x_1}\right|}{4x_1} - x_1 + 1$$

These equations define the relationships of the fixed points and their dependency on μ. We now find $x_1$and $x_5$for different values of μnumerically.



The graph shows that as the magnitude of the repulsive (positive) charge increases, the equilibrium points move farther out. When μ equals 0, the equilibrium point is that of the harmonic oscillator without an electric field which is at the equilibrium lengths of the springs. Finally, as μbecomes more negative, the masses move closer to the center and closer to each other. However, once μ falls below ˜.6, the electric force is greater than the spring force and there are no longer any equilibrium points for this case when $\mu < .6$.

Therefore, the fixed points for when $cos(x_3 - x_7) = -1$ are:

$$X_1 \& X_5\ (defined\ above)\ (1)$$

$$X_2 = X_4 = X_6 = X_8 = 0\ (2)$$

$$X_3 = X_7 + \pi\ (3)$$

Essentially, this definition of the fixed point encompasses an open set of fixed points around the circle given by (3).

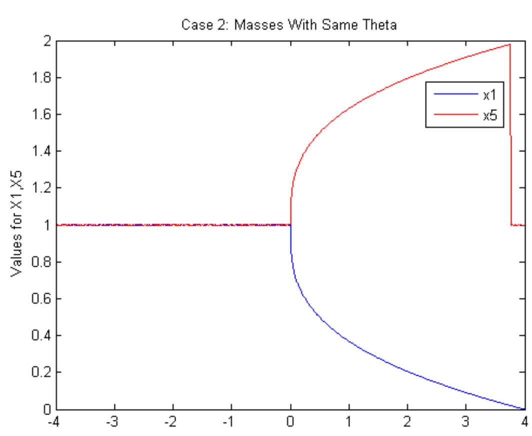
Case 2

In our second case, we will examine when the masses have the same value for X_3 and X_7. Therefore, $cos(X_3 - X_7) = 1$. We find that $x_1 = 2 - x_5$ and get the following two equations after plugging in $x_1 = x_5$:

$$0 = \frac{-\mu\left|\frac{1}{x_5-1}\right|}{4(x_5-1)} + x_5 - 1$$

$$0 = \frac{\mu\left|\frac{1}{x_1-1}\right|}{4(x_1-1)} - x_1 + 1$$

These equations define the relationships of the fixed points and their dependency on μ. We now find $x_1$and $x_5$for different values of μnumerically.



This graph shows that as μ becomes larger and positive, there is a point where the repulsion pushes one of the masses into the center which is a singularity. While the model does not capture what occurs when the repulsion pushes one of the masses into the center of the orbits, the physical analog would never allow a mass to enter the center of the orbit due to the presence of a physical spring. Therefore, we would qualify this mathematical model as moderately accurate of the physical model. It is also important to notice, that there is another fixed point by symmetry that would exist in the case where the positions of $X_1$and X_5 were reversed. This symmetry in the model allows us to examine one case.

When μ equals zero, the system has a fixed point corresponding to the uncoupled mechanical system at $X_1 = 1,\ X_5 = 1$. Finally, as μbecomes negative, the fixed points vanish. This result is what we would expect because as the charges become attractive, the masses will attract each other and approach each other for an infinite amount of time, never reaching a fixed point due to there being a discontinuity in the vector field when $X_1 = X_5$.

Therefore, the fixed points for when $cos(x_3 - x_7) = 1$ are:

$$X_1 \& X_5 \ (defined\ above)\ (1)$$

$$X_2 = X_4 = X_6 = X_8 = 0\ (2)$$

$$X_3 = X_7\ (3)$$

Essentially, this definition of the fixed point encompasses an open set of fixed points around the circle given by (3).

Conclusions of Fixed-Point Identification

Combining the results of these cases, we can construct a bifurcation diagram where the number of fixed points varies as a function of μ. This bifurcation can be summarized in the following:

$$\mu > 4 :\ \ 1\ fixed\ point$$

$$0 \leq \mu < 4 :\ \ 2\ fixed\ points$$

$$-.6 < \mu < 0 :\ \ 1\ fixed\ point$$

$$\mu < -.6 :\ \ No\ fixed\ points$$

Now with an understanding of the existence of fixed points in our system, we may move on to characterize them by examining the solutions around them.

Fixed-Point Characterization

First, we must acknowledge that our system operates in eight dimensions. From two-dimensional analogs, we know there are certain possibilities for characterizing fixed point within this domain. We can use this knowledge to characterize these fixed points in eight dimensions. Since our system is conservative, we know that these fixed points can have behaviors similar only to centers and saddles. We rule out unstable fixed points as time-reversibility turns unstable fixed points into stable fixed points. Since a conservative cannot have stable fixed points, we rule out both unstable and stable fixed points. Therefore, our system can only exhibit saddle-like or center-like behavior.

We will characterize saddle-like behavior as a fixed point with both positive and negative eigenvalues, and we will characterize center-like behavior as a fixed point with at least two purely imaginary eigenvalues. We imagine that with more dimensions, fixed points can display multiple behaviors. In conducting this analysis, we noticed that our system displayed both types of behavior. Therefore, we came to the conclusion that these fixed points were a combination of saddles and centers due to their ability to contain multiple sets of eigenvalues. Also, since our fixed points are really an open set of fixed points (defined by (3) above), they are also non-isolated fixed points in these dimensions. The physical explanation of this is the two masses can be at equilibrium at any θaround the circle. We also noticed that for different values of μ, different types of fixed points could exist. For the purpose of this analysis, we will cluster the group of non-isolated fixed points as one type of fixed point and therefore, we will examine various values of μ and the corresponding nature of the fixed points.

Case 1:

$$0 = \frac{\mu\left|\frac{1}{x_5}\right|}{4x_5} - x_5 + 1$$

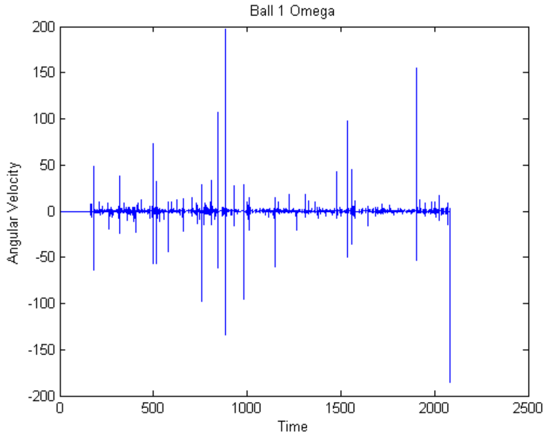
$$0 = \frac{\mu\left|\frac{1}{x_1}\right|}{4x_1} - x_1 + 1$$

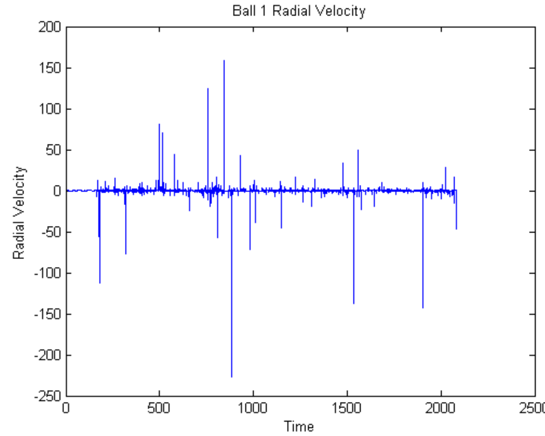
$$X_2 = X_4 = X_6 = X_8 = 0$$

$$X_3 = X_7 + \pi$$

$-.6 < \mu < 0$

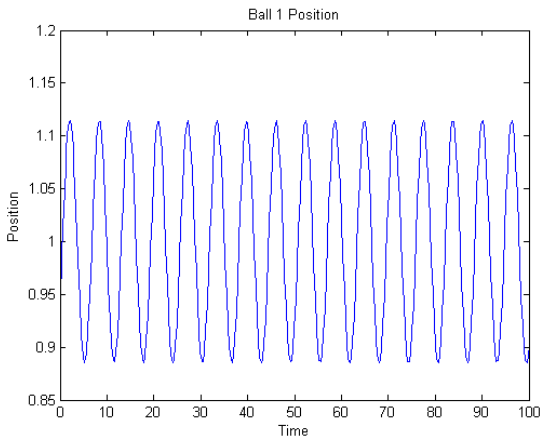
In first studying the solutions around this fixed point for the given range of μ, we discovered that solutions displayed a behavior such that for any initial conditions, the long term behavior of the system approached infinity. This can be seen in the example where the masses are slightly perturbed from fixed point (below). The physical explanation for what is occurring is that due to the attractive force between the masses, as they approach each other, their velocities become infinitely positive as their potentials become infinitely negative. Eventually, for all initial conditions, there is eventually a "collision" of the masses and their radial/angular velocities approach infinity. It was noticed that as $|\mu|$ became smaller, the longer it took before the collision occured. In our numerical simulations, this approach towards infinity causes the ODE solver in matlab to crash leaving us with a pre-mature ending to our simulation. Therefore, we can characterize this fixed point as exhibiting non-linear saddle-like behavior for the given range of μ.

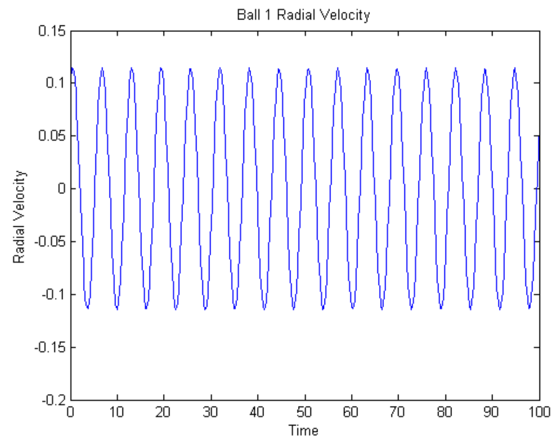




$\mu = 0$

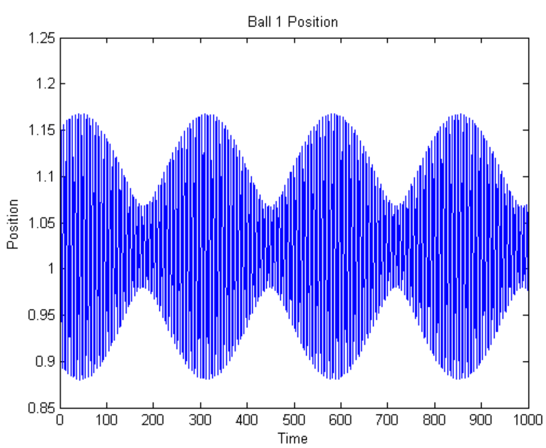
When we uncouple the dynamics of the two masses ($\mu = 0$), we are left with the simple case where the fixed point simply a center and shows completely oscillatory behavior.

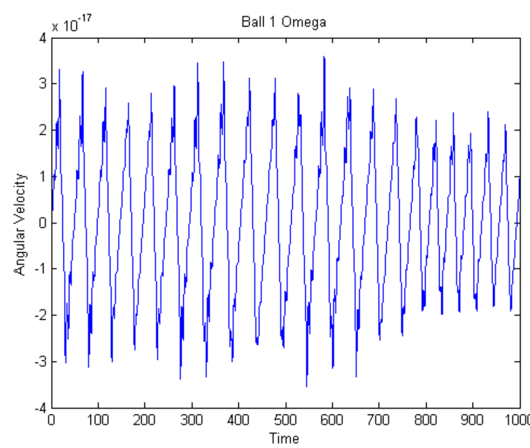


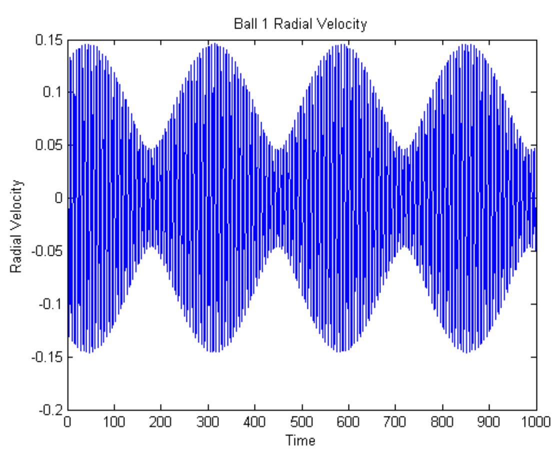


$\mu > 0$

When we introduce repulsive charges into our system $\mu > 0$, we begin to see quasi-periodic behavior around the fixed point. However, we completely lose our ability to obtain unbounded solutions because "collisions" of the masses are no longer possible. Therefore, this fixed point can be characterized as exhibiting center-like behavior.







Case 2:

$$0 = \frac{-\mu\left|\frac{1}{x_5-1}\right|}{4(x_5-1)} + x_5 - 1$$

$$0 = \frac{\mu\left|\frac{1}{x_1-1}\right|}{4(x_1-1)} - x_1 + 1$$

$$X_2 = X_4 = X_6 = X_8 = 0$$

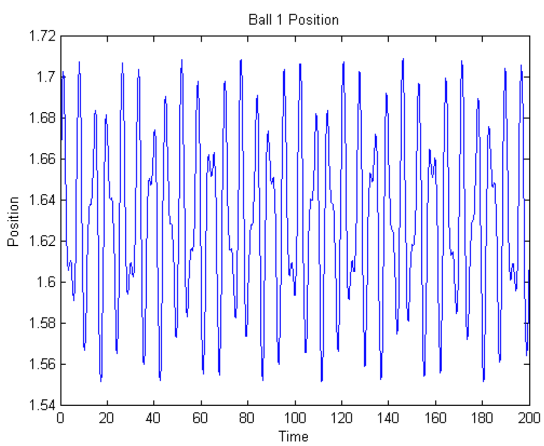
$$X_3 = X_7$$

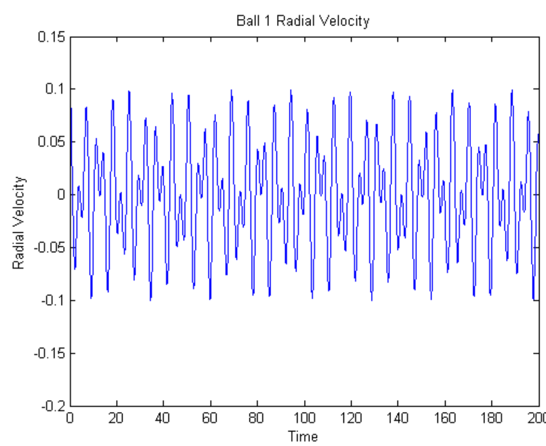
$\mu = 0$

When we uncouple the dynamics of the two masses ($\mu = 0$), we are left with the simple case where the fixed point is simply a center and shows completely oscillatory behavior. There is no need to show results from simulations as they are identical to the other fixed point examined previously.

$0 < \mu < 4$

When we introduce repulsive charges into our system $\mu > 0$, we begin to see quasi-periodic behavior around the fixed point. Therefore, by the same reasoning as before, we know this system cannot have unbounded solutions and therefore cannot exhibit saddle-like behavior. Therefore, this fixed point for the given values of μ must exhibit center-like behavior.





Fixed-Point Conclusions

After analyzing our system, the results of our fixed-point analysis have revealed that with the variation of the parameter μ , our system can display bifurcations where fixed-points appear and reappear as well as change their characteristic behavior. These results can be summarized by the following:

- $\mu < -.6$: There are no fixed points in the system as the attractive charge in the physical system dominates the behavior of the system.
- $-.6 < \mu < 0$: There is one fixed point in the system that exhibits saddle-like behavior. The physical representation is of the two charged masses being an oppososite sides of the circle they orbit where the spring forces are able to balance the attractive forces.
- $0 \leq \mu < 4$: There are two fixed points in the system that exhibit center-like behavior. The physical representation is of the two charged masses either being on opposite sides of the circle or on the same side. In either case, their repulsion is balanced by the spring forces.
- $\mu > 4$: There is one fixed point in the system that exhibits center-like behavior. The physical representation is of the two charged masses being on opposite sides of the circle they orbit where the spring forces are able to balance the repulsive forces.

Chaos

In studying this system, we noticed that by adding an electric field, we allowed for the possibility of chaos and for strange attractors.

We define chaos as the following (Strogatz 1985):

- Aperiodic long-term behavior
- Deterministic, meaning no random or noisy inputs or parameters.
- Sensitive Dependence on Initial Conditions

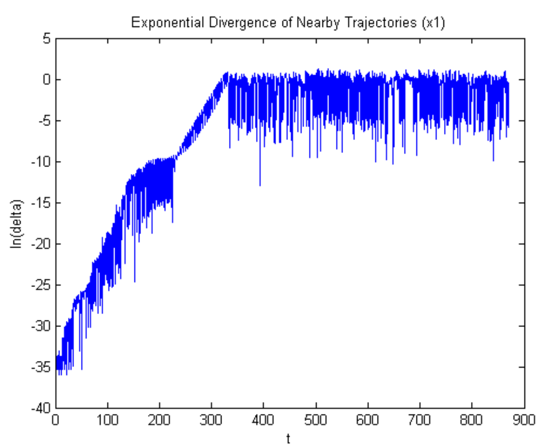
We define an attractor as the following (Strogatz 1985):

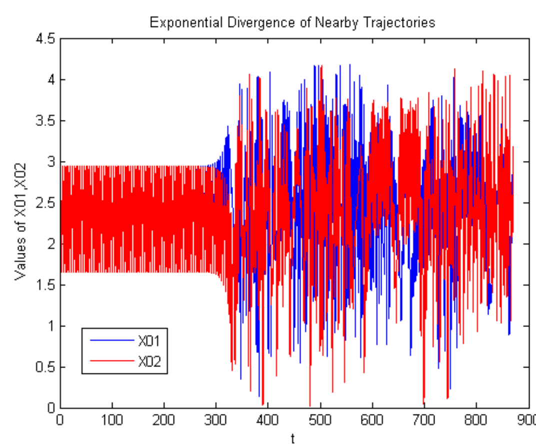
- A is an invariant set: any trajectory x(t) that starts in A stays in A for all time.
- A attracts an open set of initial conditions.
- A is minimal: there is no proper subset of A that satisfies conditions 1 and 2.
- Finally, we define a strange attractor as an attractor that exhibits sensitive dependence on initial conditions.

In order to analyze this, we decided to study different cases for μ. To analyze the presence of chaos, we ran simulations of a given initial condition with a slight perturbation (10^{-15}) , and then monitored the difference between the solutions over time to see if an exponential divergence occured (indicating a sensitive dependence on initial conditions) and whether there was aperiodic long-term behavior. We assume that our system is deterministic. In running these simulations, the first thing we learned was in choosing an initial condition to study, we must choose one that is sufficiently far away from any possible fixed points. The reason for this is if we are sufficiently close to a fixed point, the solution will not exhibit chaos but rather adopt the local behavior of the fixed point.

$\mu < 0$

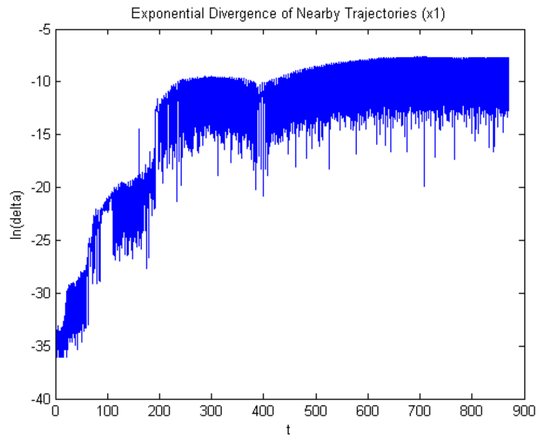
In analyzing the system for the case where there are attractive charges, simulations show that for initial conditions away from any fixed points, chaos does occur. However, while there is a sensitive dependence on initial conditions, since all solutions tend to infinity as the particles eventually "collide," we know that there is no strange attractor in the system for $\mu < 0$.

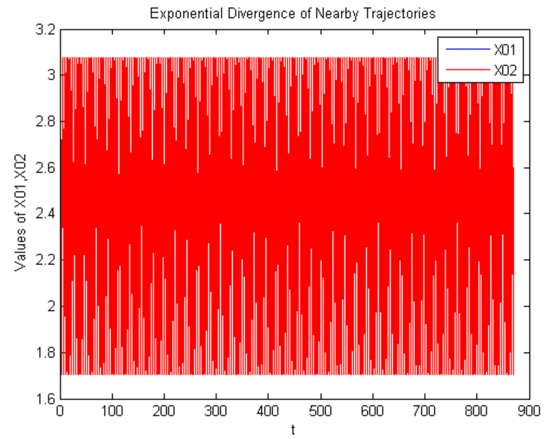




$\mu = 0$

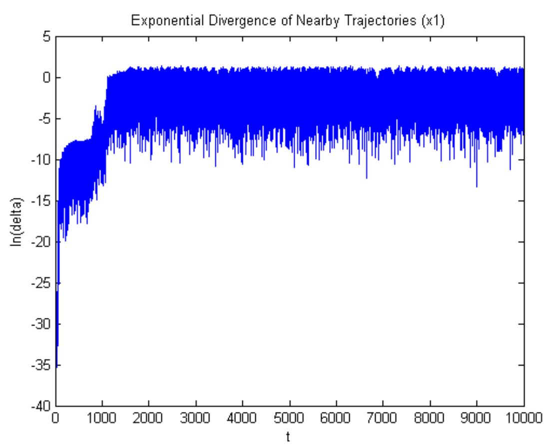
In analyzing the system for the case where there are no charges, simulations show that for initial conditions away from any fixed points, while there is a sensitive dependence on initial conditions, there is no long term aperiodic behavior. We can come to the conclusion that this system does not exhibit chaos based on long term simulations showing periodic predictable solutions. Finally, we notice that all solutions are bounded in phase-space, and therefore, according to the defined criteria above, we know there must be a strange attractor in the system.

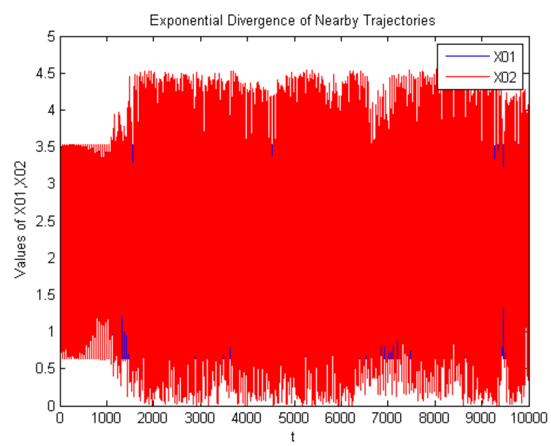




$\mu > 0$

In analyzing the system for the case where there are repulsive charges, simulations show that for initial conditions away from any fixed points, chaos does occur. Since all solutions are bounded for all time, we know there must be an attractor that appears when $\mu > 0$. Furthermore, since there is a sensitive dependence on initial conditions, we can classify the attractor as a strange attractor. However, it is interesting to note in this analysis, the chaos does not seem to be as severe as in the case where $\mu < 0$ and attractive charges are present. In the case where there are repulsive charges, the time horizon is much longer than in the case of where there are attractive charges.





Chaos Conclusions

In studying this system, we can summarize our findings about chaos and attractors in the system as the following:

- $\mu < 0$: There is chaotic behavior in the system.
- $\mu = 0$: There is no chaotic behavior but there exists a strange attractor in phase-space.
- $\mu > 0$: There is a combination of chaotic behavior as well as a strange attractor.

Future Steps & Conclusions

In studying this system, we realized the difficulty in analyzing a system in 8 dimensions. Therefore, much of our conclusions were based on numerical simulations rather than analytical techniques. In the future, we believe a better analytical analysis could be completed through linearization of the system and determining the eigenvalues of the system. If this was completed, we would be able to validate our conclusions about the fixed point characterization through the use of eigenvalues rather than numerical simulations. Another approach to simplifying the system would have been to analyze the system using our reference frame as one of the masses. In this manner, we would have been able to reduce our equations in half. This would allow for a more feasible manner to determine the fixed points and characterize them.

In conclusion, our team analyzed a system of electrically coupled harmonic oscillators. Through starting with the basic laws of physics, we were able to validate our initial dimensional differential equations through conservation of energy and momentum laws. Once this was completed, we completed a dimensional analysis to identify important parameters in the behavior of the system and decided to study what we believed was the most influential parameter in the system's qualitative behavior. We termed this parameter, μ, and it represented the ratio of the electrical forces to the spring forces in the system. By examining the range of values μ could take, we identified bifurcations of fixed points depending on the sign and magnitude of μ. Once we identified the fixed points, we moved on to characterizing them through numerical perturbation studies around the fixed points. Finally, we examined the possibility for chaos and strange attractors in our system. We found that once again, the parameter μ influenced the existence and behaviors of chaotic solutions and strange attractors. Through a comprehensive analysis of the mathematical system, we can say that it is evident that in the physical system, the strongest influence on the behavior of the coupled oscillators is whether the electric forces or spring forces dominate the behavior of the system. This parameter not only determines fixed points, but also the behavior around these fixed points and the overall presence of chaotic solutions as well as the existence of strange attractors.

References

Hirsch, Morris W., Stephen Smale, and Robert L. Devaney. Differential Equations, Dynamical Systems, and an Introduction to Chaos. San Diego, CA: Academic, 2004. Print.

MATLAB Code

```
%The following code was used to analyze the dimensional system.
function main(tim,ch,range)

K1=10; K2=10; q1=1e-6; q2=1e-6; m1=1; m2=1;R1=1;R2=1;Ke= 8.987e9;
options=odeset('RelTol',1e-12,'Stats','on');
Xo=[1;0;pi;-1;1;0;0;-.5];
tspan=[0,tim];

tic
[t,X]=ode113(@Testfunction,tspan,Xo,options);
toc

[X3,X1]=pol2cart(X(:,3),X(:,1));
[X7,X5]=pol2cart(X(:,7),X(:,5));

if ch==1
p1=plot(X3,X1);
set(p1,'Color','blue');
hold on;
p2=plot(X7,X5);
set(p2,'Color','red');

elseif ch==2;
    for j=1:5:size(X(:,1),1);
    p1=plot(X3(1:j),X1(1:j));
    hold on;
    set(p1,'Color','blue');
    hold on;
    p2=plot(X7(1:j),X5(1:j));
    set(p2,'Color','red');
     M(j)=getframe;
    end
%
numtimes=3;
fps=10;
movie(M,numtimes,fps)


elseif ch==3;
dis=(sqrt(power(X(:,1),2)-2.*X(:,1).*X(:,5).*cos(X(:,3)-X(:,7))+power(X(:,5),2)));
    eb1=(1/2)*m1.*power(X(:,1),2).*power(X(:,4),2)+(1/2)*m1.*power(X(:,2),2)+(1/2)*K1
.*power(X(:,1)-R1,2)+(Ke*q1*q2)
./(sqrt(power(X(:,1),2)-2.*X(:,1).*X(:,5).*cos(X(:,3)-X(:,7))+power(X(:,5),2)));
    p2=plot(t,eb1,'*');
    set(p2,'Color','c');
    hold on;
    eb2=(1/2)*m2.*power(X(:,5),2).*power(X(:,8),2)+(1/2)*m2.*power(X(:,6),2)+(1/2)*K2
.*power(X(:,5)-R2,2)+(Ke*q1*q2)
./(sqrt(power(X(:,1),2)-2.*X(:,1).*X(:,5).*cos(X(:,3)-X(:,7))+power(X(:,5),2)));
    p3=plot(t,eb2);
    set(p3,'Color','b');
    hold on;
    total=eb1+eb2;
    p1=plot(t,total);
    maxi=max(total);
    set(p1,'Color','red');
    axis([0 tim 0 (1.1)*maxi]);
    xlabel('Time');
    ylabel('Energy');
```

```
legend('Energy Ball 1','Energy Ball 2','Energy Total');

figure
eb1rot=(1/2)*m1.*power(X(:,1),2).*power(X(:,4),2);
eb1trans=(1/2)*m1*power(X(:,2),2);
eb1spring=(1/2)*K1.*power(X(:,1)-R1,2);
eb1elec=(Ke*q1*q2)./dis;
p5=plot(t,eb1rot);
hold on;
set(p5,'Color','red');
p6=plot(t,eb1trans);
hold on;
set(p6,'Color','blue');
p7=plot(t,eb1spring);
hold on;
set(p7,'Color','c');
hold on;
p9=plot(t,eb1elec);
set(p9,'Color','m');
hold on;
p8=plot(t,eb1rot+eb1spring+eb1trans+eb1elec);
hold on;
set(p8,'Color','black');
legend('Rotational Energy','Translational','Spring Energy','Electrical Energy','Total Energy');
title('Ball 1 Energy');
axis([0 tim 0 (1.1)*max(eb1rot+eb1spring+eb1trans+eb1elec)])
 xlabel('Time (s)');
ylabel('Energy (J)');

figure
eb2rot=(1/2)*m2.*power(X(:,5),2).*power(X(:,8),2);
eb2trans=(1/2)*m2*power(X(:,6),2);
eb2spring=(1/2)*K2.*power(X(:,5)-R2,2);
eb2elec=(Ke*q1*q2)./dis;
p5=plot(t,eb2rot);
hold on;
set(p5,'Color','red');
p6=plot(t,eb2trans);
hold on;
set(p6,'Color','blue');
p7=plot(t,eb2spring);
hold on;
set(p7,'Color','c');
hold on;
p9=plot(t,eb2elec);
set(p9,'Color','m');
hold on;
p8=plot(t,eb2rot+eb2spring+eb2trans+eb2elec);
hold on;
set(p8,'Color','black');
legend('Rotational Energy','Translational','Spring Energy','Electrical Energy','Total Energy');
title('Ball 2 Energy');
axis([0 tim 0 (1.1)*max(eb2rot+eb2spring+eb2trans+eb2elec)])
xlabel('Time (s)');
ylabel('Energy (J)');

figure
eball1=eb1rot+eb1spring+eb1trans+eb1elec;
eball2=eb2rot+eb2spring+eb2trans+eb2elec;
p1=plot(t,eball1);
set(p1,'Color','blue');
```

```
    hold on;
    p2=plot(t,eball2);
    set(p2,'Color','red');
    hold on;
    p3=plot(t,eball1+eball2);
    set(p3,'Color','black');
    legend('Ball 1','Ball 2','Total');
    title('Total Energy within the System');
     xlabel('Time (s)');
    ylabel('Energy (J)');


    elseif ch==5;

        p1=plot(t,X(:,4));
        title('ang speed ball 1')
        set(p1,'Color','red');
        hold on;
        p4=plot(t,X(:,8));
        set(p4,'Color','blue');
        title('ang speed ball 2');
        figure
        p2=plot(t,X(:,2));
        title('trans speed ball 1');
        hold on;
        p3=plot(t,X(:,6));
        title('trans speed ball 2');
        set(p2,'Color','blue');
        set(p3,'Color','red');



elseif ch==6;

    %Calculate angular Momentum of Ball 1
    angmom1=m1*power(X(:,1),2).*X(:,4);
    linmom1=m1*X(:,2);

    plot(t,angmom1);
    title('Angular Momentum of Ball 1');
    figure
    plot(t,linmom1);
    title('Linear Momentum of Ball 1');

    Calculate the angular Momentum of Ball 2
    angmom2=m2*power(X(:,5),2).*X(:,8);
    linmom2=m2*X(:,6);

    figure
    plot(t,angmom2);
    title('Angular Momentum of Ball 2');
    figure
    plot(t,linmom2);
    title('Linear Momentum of Ball 2');

    %Calculate the Total Linear Momentum
%
    figure
    p1=plot(t,linmom1);
    set(p1,'Color','blue');
    hold on;
```

```
    p2=plot(t,linmom2);
    set(p2,'Color','red');
    hold on;
    p3=plot(t,linmom1+linmom2);
    set(p3,'Color','c');
    title('Linear Momentum');
    legend('Ball 1', 'Ball 2','Total');

    %Calculate the Total Angular Momentum

    figure
    p1=plot(t,angmom1);
    set(p1,'Color','blue');
    hold on;
    p2=plot(t,angmom2);
    set(p2,'Color','red');
    hold on;
    p3=plot(t,angmom1+angmom2);
    set(p3,'Color','c');
    title('Angular Momentum');
    legend('Ball 1', 'Ball 2','Total');
    xlabel('Time (s)');
    ylabel('Momentum (N*m*s)');



else

end

return


%Dimensional ODE's
function [dx_dt]=Testfunction(t,x);

K1=10; K2=10; q1=1e-6; q2=1e-6; m1=1; m2=1;R1=1;R2=1;Ke= 8.987e9;
C=Ke*q1*q2;
A1=((x(1)-x(5)*cos(x(3)-x(7))))/((x(1)^2-2*x(1)*x(5)*cos(x(3)-x(7))+x(5)^2)^(3/2));
A2=((x(5)-x(1)*cos(x(3)-x(7))))/((x(1)^2-2*x(1)*x(5)*cos(x(3)-x(7))+x(5)^2)^(3/2));
B1=(x(5)*x(1)*sin(x(3)-x(7)))/((x(1)^2-2*x(1)*x(5)*cos(x(3)-x(7))+x(5)^2)^(3/2));
B2=(x(5)*x(1)*sin(x(7)-x(3)))/((x(1)^2-2*x(1)*x(5)*cos(x(3)-x(7))+x(5)^2)^(3/2));

dx_dt(1)=x(2);
dx_dt(2)=(C/(m1))*A1+x(1)*(x(4)^2)-(K1*(x(1)-R1))/m1;
dx_dt(3)=x(4);
dx_dt(4)=(C/(m1*(x(1)^2)))*B1-(2*x(4)*x(2))/x(1);
dx_dt(5)=x(6);
dx_dt(6)=(C/m2)*A2+x(5)*(x(8)^2)-(K2*(x(5)-R2))/m2;
dx_dt(7)=x(8);
dx_dt(8)=(C/(m2*(x(5)^2)))*B2-(2*x(8)*x(6))/x(5);

dx_dt=transpose(dx_dt);

return

%The following code was used to analyze the dimensionless system.
function dimensionless(tim,ch,range);

mu=1;delta=1;lambda=1;zeta=1;

options=odeset('RelTol',1e-12,'Stats','on');
```

```
%x1=.943877; x5=x1;
x1=power(2,1/3)/2+1; x5=1-power(2,1/3)/2;
Xo=[x1;.1;0;0;x5;0;0;0];
tspan=[0,tim];

tic
[t,X]=ode113(@dimensionlessode,tspan,Xo,options);
toc

[X3,X1]=pol2cart(X(:,3),X(:,1));
[X4,X2]=pol2cart(X(:,4),X(:,2));
[X7,X5]=pol2cart(X(:,7),X(:,5));
[X8,X6]=pol2cart(X(:,8),X(:,6));

if ch==1
p1=plot(X3,X1);
set(p1,'Color','blue');
hold on;
p2=plot(X7,X5);
set(p2,'Color','red')
title('Trajectories of Masses');
xlabel('X-Position');
ylabel('Y-Position');

elseif ch==2;
     for j=1:5:size(X(:,1),1);
        p1=plot(X3(1:j),X1(1:j));
        hold on;
        set(p1,'Color','blue');
        hold on;
        p2=plot(X7(1:j),X5(1:j));
        set(p2,'Color','red');
        M(j)=getframe;
    end

numtimes=3;
fps=10;
movie(M,numtimes,fps)

elseif ch==3;

    dis=(sqrt(power(X(:,1),2)-2.*X(:,1).*X(:,5).*cos(X(:,3)-X(:,7))+power(X(:,5),2)));

    %Plot Energy Profile for Ball 1
     figure
    eb1rot=power(X(:,1),2).*power(X(:,4),2);
    eb1trans=power(X(:,2),2);
    eb1spring=power(X(:,1)-1,2);
    eb1elec=(2*mu)./dis;
    p5=plot(t,eb1rot);
    hold on;
    set(p5,'Color','red');
    p6=plot(t,eb1trans);
    hold on;
    set(p6,'Color','blue');
    p7=plot(t,eb1spring);
    hold on;
    set(p7,'Color','c');
    hold on;
    p9=plot(t,eb1elec);
    set(p9,'Color','m');
```

```
    hold on;
    p8=plot(t,eb1rot+eb1spring+eb1trans+eb1elec);
    hold on;
    set(p8,'Color','black');
    legend('Rotational Energy','Translational','Spring Energy','Electrical Energy','Total Energy');
    title('Ball 1 Energy');
    axis([0 tim 0 (1.1)*max(eb1rot+eb1spring+eb1trans+eb1elec)])

    %Plot Energy for Ball 2
     figure
    eb2rot=(power(X(:,5),2).*power(X(:,8),2))/lambda;
    eb2trans=(power(X(:,6),2))/lambda;
    eb2spring=(power(X(:,5)-(1/delta),2))/zeta;
    eb2elec=(2*mu)./dis;
    p5=plot(t,eb2rot);
    hold on;
    set(p5,'Color','red');
    p6=plot(t,eb2trans);
    hold on;
    set(p6,'Color','blue');
    p7=plot(t,eb2spring);
    hold on;
    set(p7,'Color','c');
    hold on;
    p9=plot(t,eb2elec);
    set(p9,'Color','m');
    hold on;
    p8=plot(t,eb2rot+eb2spring+eb2trans+eb2elec);
    hold on;
    set(p8,'Color','black');
    legend('Rotational Energy','Translational','Spring Energy','Electrical Energy','Total Energy');
    title('Ball 2 Energy');
    axis([0 tim 0 (1.1)*max(eb2rot+eb2spring+eb2trans+eb2elec)])

    figure
    eball1=eb1rot+eb1spring+eb1trans+eb1elec;
    eball2=eb2rot+eb2spring+eb2trans+eb2elec;
    p1=plot(t,eball1);
    set(p1,'Color','blue');
    hold on;
    p2=plot(t,eball2);
    set(p2,'Color','red');
    hold on;
    p3=plot(t,eball1+eball2);
    set(p3,'Color','black');
    legend('Ball 1','Ball 2','Total');
    title('Total Energy in System');

elseif ch==4;

     angmom1=power(X(:,1),2).*X(:,4);
     angmom2=(power(X(:,5),2).*X(:,8))/lambda;

     figure
    p1=plot(t,angmom1);
    set(p1,'Color','blue');
    hold on;
    p2=plot(t,angmom2);
    set(p2,'Color','red');
    hold on;
    p3=plot(t,angmom1+angmom2);
```

```
    set(p3,'Color','c');
    title('Angular Momentum');
    legend('Ball 1', 'Ball 2','Total');

elseif ch==5;
    figure
    p1=plot(t,X(:,1));
    title('Ball 1 Position');
    xlabel('Time');
    ylabel('Position');

    figure
    p2=plot(t,X(:,2));
    title('Ball 1 Radial Velocity');
    xlabel('Time');
    ylabel('Radial Velocity');

    figure
    p3=plot(t,X(:,3));
    title('Ball 1 Theta');

    figure
    p4=plot(t,X(:,4));
    title('Ball 1 Omega');
    xlabel('Time');
    ylabel('Angular Velocity');

else

end

return

%Dimensionless ODE's
function [dx_dt]=dimensionlessode(t,x);

mu=0;delta=1;lambda=1;zeta=1;

dis=(x(1)^2-2*x(1)*x(5)*cos(x(3)-x(7))+x(5)^2)^(3/2);
A=x(1)-x(5)*cos(x(3)-x(7));
B=x(1)*x(5)*sin(x(3)-x(7));
C=x(5)-x(1)*cos(x(3)-x(7));
D=x(1)*x(5)*sin(x(7)-x(3));


dx_dt(1)=x(2);
dx_dt(2)=mu*(A/dis)+x(1)*(x(4)^2)-(x(1)-1);
dx_dt(3)=x(4);
dx_dt(4)=(mu/((x(1))^2))*(B/dis)-(2*x(2)*x(4))/x(1);
dx_dt(5)=x(6);
dx_dt(6)=mu*lambda*(C/dis)+x(5)*((x(8))^2)-(lambda/zeta)*(x(5)-(1/delta));
dx_dt(7)=x(8);
dx_dt(8)=((mu*lambda)/((x(5))^2))*(D/dis)-(2*x(6)*x(8))/x(5);

dx_dt=transpose(dx_dt);

return

%The following code was used to analyze the presence of sensitive
%dependence on initial conditions
function chaos(tim1,tim2,k,p,tstep);
```

```
% mu=.1;delta=1;lambda=1;zeta=1;

options=odeset('RelTol',1e-13,'Stats','on');
%x1=.943877; x5=x1;
%x1=power(2,1/3)/2+1; x5=1-power(2,1/3)/2;
x1=2; x5=2;
X01=[x1;1;pi;1;x5;1;0;-1];
tspan=[0:tstep:tim2];

tic
[t1,X1]=ode113(@dimensionlessode,tspan,X01,options);
toc

X02= X01+p;
tic
[t2,X2]=ode113(@dimensionlessode,tspan,X02,options);
toc


%Calculate  liaponov exponent for x
delta=log(abs(X1(:,k)-X2(:,k)));
init=(tim1/tstep);
fin=(tim2/tstep);
plot(t1(init:fin,1),delta(init:fin,1));
xlabel('t');
ylabel('ln(delta)');
title('Exponential Divergence of Nearby Trajectories (x1)');

figure
p1=plot(t1,X1(:,k));
set(p1,'Color','blue');
hold on;
p2=plot(t1,X2(:,k));
set(p2,'Color','red');
xlabel('t');
ylabel('Values of X01,X02');
title('Exponential Divergence of Nearby Trajectories');
legend('X01','X02');


end
```